

# The Moore-Penrose inverse of differences and products of projectors in a ring with involution

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**Abstract:** In this paper, we study the Moore-Penrose inverses of differences and products of projectors in a ring with involution. Also, some necessary and sufficient conditions for the existence of such inverses are given, and their expressions are presented.

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## 1 Introduction

Throughout this paper,  $R$  is a unital  $*$ -ring, that is a ring with unity 1 and an involution  $a \mapsto a^*$  satisfying that  $(a^*)^* = a$ ,  $(a + b)^* = a^* + b^*$ ,  $(ab)^* = b^*a^*$ . Recall that an element  $a \in R$  is said to have a Moore-Penrose inverse (abbr. MP-inverse) if there exists  $b \in R$  such that the following equations hold [12]:

$$aba = a, bab = b, (ab)^* = ab, (ba)^* = ba.$$

Any  $b$  that satisfies the equations above is called a MP-inverse of  $a$ . The MP-inverse of  $a \in R$  is unique if it exists and is denoted by  $a^\dagger$ . By  $R^\dagger$  we denote the set of all MP-invertible elements in  $R$ .

MP-inverse of differences and products of projectors in various sets attracts wide attention from many scholars. For instance, Cheng and Tian [1] studied the MP-inverses of  $pq$  and  $p - q$ , where  $p, q$  are projectors in complex matrices. Li [10] investigated how

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to express MP-inverses of product  $pq$  and differences  $p - q$  and  $pq - qp$ , for two given projectors  $p$  and  $q$  in a  $C^*$ -algebra. Later, Deng and Wei [3] derived some formulae for the MP-inverse of the differences and the products of projectors in a Hilbert space. Recently, Zhang et al. [13] obtained the equivalences for the existences of differences and products of projectors in a  $*$ -reducing ring. More results on MP-inverses can be found in [7, 8, 12].

Motivated by [9], we investigate the equivalences for the existences of the MP-inverse of differences and products of projectors in a ring with involution. Moreover, the expressions of the MP-inverse of differences and products of projectors are presented. Some known results in  $C^*$ -algebras are extended.

Note that neither dimensional analysis nor special decomposition in Hilbert spaces and  $C^*$ -algebras can be used in rings. The results in this paper are proved by a purely ring theoretical method.

## 2 Some lemmas

We begin with some lemmas which play an important role in the sequel.

In 1992, Harte and Mbekhta [5] showed the excellent result in  $C^*$ -algebras that if  $a$  is MP-invertible, then  $a^*c = ca^*$  and  $ac = ca$  imply  $a^\dagger c = ca^\dagger$ . More precisely, it follows from [11, Corollary 12] that in a  $*$ -semigroup  $a^\dagger \in \text{comm}^2\{a, a^*\}$ , i.e.,  $a^\dagger$  double commutes with  $a$  and  $a^*$ . In 2013, Drazin [4] then proved the following.

**Lemma 2.1.** [4, Corollary 2.7] *Let  $S$  be any  $*$ -semigroup, let  $a_1, a_2, d \in S$ , and suppose that  $a_1$  and  $a_2$  each have Moore-Penrose inverses  $a_1^\dagger, a_2^\dagger$ , respectively. Then, for any  $d \in S$ ,  $da_1 = a_2d$  and  $da_1^* = a_2^*d$  together imply  $a_2^\dagger d = da_1^\dagger$ .*

The following result in  $C^*$ -algebras was considered by Koliha [6]. For the convenience of the reader, we give its proof in a ring.

**Lemma 2.2.** *Let  $a, b \in R^\dagger$  with  $ab = ba$  and  $a^*b = ba^*$ . Then  $ab \in R^\dagger$  and  $(ab)^\dagger = b^\dagger a^\dagger = a^\dagger b^\dagger$ .*

*Proof.* It follows from Lemma 2.1 that  $a^\dagger b = ba^\dagger$  and  $b^\dagger a = ab^\dagger$ . As  $b^*a = ab^*$  and  $b^*a^* = a^*b^*$ , then  $b^*a^\dagger = a^\dagger b^*$ , which together with  $ba^\dagger = a^\dagger b$  imply  $a^\dagger b^\dagger = b^\dagger a^\dagger$ . Note

that  $aa^\dagger$  commutes with  $b$  and  $b^\dagger$ . Also,  $bb^\dagger$  commutes with  $a$  and  $a^\dagger$ . Hence,  $b^\dagger a^\dagger$  satisfies four equations of Penrose. Indeed, we have

$$(i) (abb^\dagger a^\dagger)^* = (aba^\dagger b^\dagger)^* = (aa^\dagger bb^\dagger)^* = bb^\dagger aa^\dagger = aa^\dagger bb^\dagger = aba^\dagger b^\dagger = abb^\dagger a^\dagger.$$

$$(ii) (b^\dagger a^\dagger ab)^* = (b^\dagger ba^\dagger a)^* = a^\dagger ab^\dagger b = b^\dagger a^\dagger ab.$$

$$(iii) abb^\dagger a^\dagger ab = aa^\dagger bb^\dagger ab = aa^\dagger bb^\dagger ba = aa^\dagger ba = aa^\dagger ab = ab.$$

$$(iv) b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger ba^\dagger ab^\dagger a^\dagger = b^\dagger ba^\dagger aa^\dagger b^\dagger = b^\dagger ba^\dagger b^\dagger = b^\dagger a^\dagger.$$

Therefore,  $ab \in R^\dagger$  and  $(ab)^\dagger = b^\dagger a^\dagger = a^\dagger b^\dagger$ .  $\square$

Penrose [12, p. 408] presented the MP-inverse of  $A + B$ , where  $A$  and  $B$  are complex matrices such that  $A^*B = 0$  and  $AB^* = 0$ . His formula indeed holds in a ring with involution.

**Lemma 2.3.** *Let  $a, b \in R^\dagger$  such that  $a^*b = ab^* = 0$ . Then  $(a + b)^\dagger = a^\dagger + b^\dagger$ .*

### 3 Main results

We say that an element  $p$  is a projector if  $p^2 = p = p^*$ . Throughout this paper, the elements  $p, q$  are projectors from the ring  $R$ .

**Theorem 3.1.** *Let  $a, b \in R^\dagger$  with  $a^*p = pa^*$  and  $b^*p = pb^*$ . Then  $ap + b(1 - p) \in R^\dagger$  and  $(ap + b(1 - p))^\dagger = a^\dagger p + b^\dagger(1 - p)$ .*

*Proof.* As  $a^*p = pa^*$ , then  $ap = pa$  since  $p$  is a projector. Similarly,  $bp = pb$ . We have  $(ap)^*b(1 - p) = 0$ . Indeed,  $(ap)^*b(1 - p) = pa^*(1 - p)b = a^*p(1 - p)b = 0$ . Also,  $ap(b(1 - p))^* = 0$ . By Lemma 2.2, it follows that  $(ap)^\dagger = a^\dagger p$  and  $(b(1 - p))^\dagger = b^\dagger(1 - p)$ . In view of Lemma 2.3, we obtain  $ap + b(1 - p) \in R^\dagger$  and  $(ap + b(1 - p))^\dagger = a^\dagger p + b^\dagger(1 - p)$ .  $\square$

Recall from [8] that an element  $a \in R$  is  $*$ -cancellable if  $a^*ax = 0$  implies  $ax = 0$  and  $xaa^* = 0$  implies  $xa = 0$ . In a  $C^*$ -algebra, every element is  $*$ -cancellable. A ring  $R$  is called  $*$ -reducing ring if all elements in  $R$  are  $*$ -cancellable.

We get the following result, under the condition of  $*$ -cancellabilities of some elements, rather than  $*$ -reducing rings in [13].

**Proposition 3.2.** *Let  $p(1-q)$  and  $q(1-p)$  be  $*$ -cancellable. Then the following conditions are equivalent:*

- (1)  $1 - pq \in R^\dagger$ , (2)  $1 - pqp \in R^\dagger$ , (3)  $p - pqp \in R^\dagger$ , (4)  $p - pq \in R^\dagger$ , (5)  $p - qp \in R^\dagger$ ,
- (6)  $1 - qp \in R^\dagger$ , (7)  $1 - qpq \in R^\dagger$ , (8)  $q - qpq \in R^\dagger$ , (9)  $q - qp \in R^\dagger$ , (10)  $q - pq \in R^\dagger$ .

*Proof.* As  $a \in R^\dagger \Leftrightarrow a^* \in R^\dagger$ , then (1)  $\Leftrightarrow$  (6) and (4)  $\Leftrightarrow$  (5). Also, as  $p$  and  $q$  play symmetric roles and (1)  $\Leftrightarrow$  (2) by [13, Theorem 4], it is then sufficient to prove that (2)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (4).

(2)  $\Rightarrow$  (3) Noting  $p - pqp = p(1 - pqp) = (1 - pqp)p$ , it is an immediate result of Lemma 2.2.

(3)  $\Rightarrow$  (2) Since  $1 - pqp = p(p - pqp) + 1 - p$  and  $(p - pqp)^* = p - pqp$ , it follows from Theorem 3.1 that  $1 - pqp \in R^\dagger$ .

(3)  $\Leftrightarrow$  (4) Note that  $a \in R^\dagger \Leftrightarrow aa^* \in R^\dagger$  and  $a$  is  $*$ -cancellable by [8, Theorem 5.4]. As  $p(1 - q)(p(1 - q))^* = p - pqp \in R^\dagger$  and  $p - pq$  is  $*$ -cancellable, the result follows.  $\square$

Recall that an element  $a \in R$  is normal if  $aa^* = a^*a$ . Further, if a normal element  $a$  is MP-invertible, then  $aa^\dagger = a^\dagger a$  by Lemma 2.2.

In 2004, Koliha, Rakočević and Straškraba [9] showed that  $p - q$  is nonsingular if and only if  $1 - pq$  and  $p + q - pq$  are both nonsingular, for projectors  $p, q$  in complex matrices, which is a  $*$ -cancellable ring. It is natural to consider whether the same property can be inherited to the MP-inverse in a ring with involution. The following result illustrates its possibility.

**Theorem 3.3.** *Let  $p - q$ ,  $p(1 - q)$  and  $q(1 - p)$  be  $*$ -cancellable. Then the following conditions are equivalent:*

- (1)  $p - q \in R^\dagger$ ,
- (2)  $1 - pq \in R^\dagger$ ,
- (3)  $p + q - pq \in R^\dagger$ .

*Proof.* (1)  $\Rightarrow$  (2) Note that  $p - q$  is normal. It follows from Lemma 2.2 that  $((p - q)^2)^\dagger = ((p - q)^\dagger)^2$ . As  $p(p - q)^2 = (p - q)^2p = p - pqp$ , then  $1 - pqp = (p - q)^2p + 1 - p$  and hence  $1 - pqp \in R^\dagger$  according to Theorem 3.1. So,  $1 - pq \in R^\dagger$  by [13, Theorem 4].

(2)  $\Rightarrow$  (1) By [13, Theorem 4], we know that  $1 - pq \in R^\dagger$  implies  $1 - pqp \in R^\dagger$ . Let  $\bar{p} = 1 - p$  and  $\bar{q} = 1 - q$ . Note that  $p(1 - q)$  is  $*$ -cancellable. We have  $1 - pq \in R^\dagger \Rightarrow p - pq = \bar{q} - \bar{p}\bar{q} \in R^\dagger$  by (1)  $\Rightarrow$  (4) in Proposition 3.2. Also, as  $\bar{q}(1 - \bar{p}) = p(1 - q)$  is  $*$ -cancellable, then  $\bar{q} - \bar{p}\bar{q} \in R^\dagger$  implies  $1 - \bar{q}\bar{p} \in R^\dagger$  by (10)  $\Rightarrow$  (6) in Proposition 3.2, which means  $1 - \bar{p}\bar{q} \in R^\dagger$  since  $a \in R^\dagger \Leftrightarrow a^* \in R^\dagger$ . Again, applying [13, Theorem 4], it follows that  $1 - \bar{p}\bar{q}\bar{p} \in R^\dagger$ .

Setting  $a = 1 - pqp$  and  $b = 1 - \bar{p}\bar{q}\bar{p}$ , then  $a^*p = pa^*$  and  $b^*p = pb^*$ . Since  $(p - q)^2 = ap + b(1 - p)$ , we obtain  $(p - q)^2 = (p - q)(p - q)^* \in R^\dagger$  by Theorem 3.1 and hence  $p - q \in R^\dagger$  from [8, Theorem 5.4].

(1)  $\Leftrightarrow$  (3) In (1)  $\Leftrightarrow$  (2), replacing  $p, q$  by  $1 - p, 1 - q$ , respectively.  $\square$

Next, we mainly consider the representations of the MP-inverse by aforementioned results.

**Theorem 3.4.** *Let  $p - q \in R^\dagger$ . Define  $F, G$  and  $H$  as*

$$F = p(p - q)^\dagger, G = (p - q)^\dagger p, H = (p - q)(p - q)^\dagger.$$

*Then, we have*

$$(1) F^2 = F = (p - q)^\dagger(1 - q),$$

$$(2) G^2 = G = (1 - q)(p - q)^\dagger,$$

$$(3) H^2 = H = H^*.$$

*Proof.* (1) We first prove  $F = (p - q)^\dagger(1 - q)$ .

As  $(p - q)^* = p - q$  and  $p - q \in R^\dagger$ , then  $(p - q)^2 \in R^\dagger$  by Lemma 2.2. Moreover,  $((p - q)^2)^\dagger = ((p - q)^\dagger)^2$ . Also,  $(p - q)(p - q)^\dagger = (p - q)^\dagger(p - q)$ . From  $p(p - q)^2 = (p - q)^2p$  and  $p((p - q)^2)^* = ((p - q)^2)^*p$ , we have  $p((p - q)^\dagger)^2 = ((p - q)^\dagger)^2p$  using Lemma 2.1.

Hence,

$$\begin{aligned} (p - q)^\dagger(1 - q) &= ((p - q)^\dagger)^2(p - q)(1 - q) = ((p - q)^\dagger)^2p(1 - q) \\ &= ((p - q)^\dagger)^2p(p - q) = p((p - q)^\dagger)^2(p - q) \\ &= p(p - q)^\dagger \\ &= F. \end{aligned}$$

We now show  $F^2 = F$ . Since  $p(p - q)^\dagger = (p - q)^\dagger(1 - q)$ , one can get

$$\begin{aligned}
F^2 &= (p - q)^\dagger(1 - q)p(p - q)^\dagger \\
&= (p - q)^\dagger(1 - q)(p - q)(p - q)^\dagger \\
&= p(p - q)^\dagger(p - q)(p - q)^\dagger \\
&= p(p - q)^\dagger \\
&= F.
\end{aligned}$$

(2) By  $F^* = G$ .

(3) It is trivial. □

Under the same symbol in Theorem 3.4, more relations among  $F$ ,  $G$  and  $H$  are given in the following result.

**Corollary 3.5.** *Let  $p - q \in R^\dagger$ . Then*

- (1)  $q(p - q)^\dagger = (p - q)^\dagger(1 - p)$ ,
- (2)  $qH = Hq$ ,
- (3)  $G(1 - q) = (1 - q)F$ .

*Proof.* (1) can be obtained by a similar proof of Theorem 3.4(1).

(2) Taking involution on (1), it follows that  $(1 - p)(p - q)^\dagger = (p - q)^\dagger q$  and hence

$$\begin{aligned}
qH &= q(p - q)(p - q)^\dagger = q(p - 1)(p - q)^\dagger \\
&= -q(p - q)^\dagger q = -(p - q)^\dagger(1 - p)q \\
&= -(p - q)^\dagger(q - p)q \\
&= Hq.
\end{aligned}$$

(3) We have

$$\begin{aligned}
G(1 - q) &= (p - q)^\dagger(p - q)(1 - q) = (p - q)^\dagger p(p - q) \\
&= (1 - q)(p - q)^\dagger(p - q) \\
&= (1 - q)F.
\end{aligned}$$

□

Keeping in mind the relations in Theorem 3.4 and Corollary 3.5, we give the following equalities, where  $\bar{a}$  denotes  $1 - a$ .

**Corollary 3.6.** *Let  $p - q \in R^\dagger$ . Then*

- (1)  $Fp = pG = pH = Hp$ ,
- (2)  $qHq = qH = Hq = HqH$ ,
- (3)  $\bar{q}\bar{F} = \bar{G}\bar{q} = \bar{q}\bar{F}\bar{q}$ ,
- (4)  $(p - q)^\dagger = F + G - H$ .

In general,  $p - q \in R^\dagger$  can not imply  $p + q \in R^\dagger$ . Such as, take  $R = \mathbb{Z}$  and  $1 = p = q \in R$ , then  $p - q = 0 \in R^\dagger$ , but  $p + q = 2 \notin R^\dagger$  since 2 is not invertible.

**Theorem 3.7.** *Let 2 be invertible in  $R$ . Then the following conditions are equivalent:*

- (1)  $pH = p$ ,
- (2)  $(p + q)H = (p + q)$ ,
- (3)  $p + q \in R^\dagger$  and  $(p + q)^\dagger = (p - q)^\dagger(p + q)(p - q)^\dagger$ .

*Proof.* (1)  $\Rightarrow$  (2) If  $pH = p$ , then  $qH = q$  by the symmetry of  $p$  and  $q$ . Hence  $(p + q)H = (p + q)$ .

(2)  $\Rightarrow$  (1) Note that  $H = (p - q)(p - q)^\dagger$  and  $p - q$  is normal. We have  $(p - q)H = p - q$  and  $p + q = (p + q)H = (q - p)H + 2pH = -(p - q) + 2pH$ , which implies  $2pH = 2p$ . Hence,  $pH = p$  since 2 is invertible.

(2)  $\Rightarrow$  (3) Let  $x = (p - q)^\dagger(p + q)(p - q)^\dagger$ . We prove that  $x$  is the MP-inverse of  $p + q$  by checking four equations of Penrose.

(i)  $((p + q)x)^* = (p + q)x$ . Indeed,

$$\begin{aligned}
 (p + q)x &= (p + q)(p - q)^\dagger(p + q)(p - q)^\dagger \\
 &= (p - q)^\dagger(1 - q + 1 - p)(p + q)(p - q)^\dagger \\
 &= (p - q)^\dagger(p - q)^2(p - q)^\dagger \\
 &= (p - q)(p - q)^\dagger.
 \end{aligned}$$

(ii)  $(x(p + q))^* = x(p + q)$ . By similar proof of (i), we have  $x(p + q) = (p - q)^\dagger(p - q)$ .

(iii) Note that the relations  $pH = Hp$  and  $qH = Hq$  in Corollary 3.6. Then

$$(p + q)x(p + q) = (p - q)(p - q)^\dagger(p + q)$$

$$\begin{aligned}
&= H(p+q) = (p+q)H \\
&= p+q.
\end{aligned}$$

(iv) It follows that  $x(p+q)x = (p-q)^\dagger(p+q)(p-q)^\dagger(p-q)(p-q)^\dagger = x$ .

(3)  $\Rightarrow$  (2) As  $p+q \in R^\dagger$  with  $(p+q)^\dagger = (p-q)^\dagger(p+q)(p-q)^\dagger$ , then

$$\begin{aligned}
p+q &= (p+q)(p+q)^\dagger(p+q) = (p+q)(p-q)^\dagger(p+q)(p-q)^\dagger(p+q) \\
&= (p+q)(p-q)^\dagger(p-q)^\dagger(1-q+1-p)(p+q) \\
&= (p+q)(p-q)^\dagger(p-q)^\dagger[(1-q)p + (1-p)q] \\
&= (p+q)(p-q)^\dagger(p-q)^\dagger[(p-q)p + (q-p)q] \\
&= (p+q)(p-q)^\dagger(p-q)^\dagger(p-q)p - (p+q)(p-q)^\dagger(p-q)^\dagger(p-q)q \\
&= (p+q)(p-q)^\dagger(p-q)(p-q)^\dagger p - (p+q)(p-q)^\dagger(p-q)(p-q)^\dagger q \\
&= (p+q)(p-q)^\dagger p - (p+q)(p-q)^\dagger q \\
&= (p+q)(p-q)^\dagger(p-q) \\
&= (p+q)H.
\end{aligned}$$

□

Next, we give a new necessary and sufficient condition of the existence of  $(p+q)^\dagger$ , where  $p$  and  $q$  commute.

**Theorem 3.8.** *Let  $p, q \in R$  with  $pq = qp$ . Then  $p+q \in R^\dagger$  if and only if  $1+pq \in R^\dagger$ .*

*In this case,  $(p+q)^\dagger = (1+pq)^\dagger p + q(1-p)$  and  $(1+pq)^\dagger = (p+q)^\dagger p + 1-p$ .*

*Proof.* Suppose  $p+q \in R^\dagger$ . As  $1+pq = p(p+q) + 1-p$ , then  $(1+pq)^\dagger = (p+q)^\dagger p + 1-p$  by Theorem 3.1.

Conversely, let  $x = (1+pq)^\dagger p + q(1-p)$ . We next show that  $x$  is the MP-inverse of  $p+q$ .

(i)  $[(p+q)x]^* = (p+q)x$ . We have

$$\begin{aligned}
(p+q)x &= (p+q)[(1+pq)^\dagger p + q(1-p)] \\
&= (1+pq)^\dagger p + (1+pq)^\dagger pq + q(1-p) \\
&= (1+pq)^\dagger(1+pq)p + q(1-p).
\end{aligned}$$



Hence,  $[(p+q)x]^* = (p+q)x$ .

(ii) It follows that  $[x(p+q)]^* = x(p+q)$  since  $p$  and  $q$  commute.

(iii)  $(p+q)x(p+q) = p+q$ . Indeed,

$$\begin{aligned}
(p+q)x(p+q) &= (p+q)[(1+pq)^\dagger(1+pq)p + q(1-p)] \\
&= (1+pq)^\dagger(1+pq)p + (1+pq)^\dagger(1+pq)pq + q(1-p) \\
&= (1+pq)^\dagger(1+pq)p(1+pq) + q(1-pq) \\
&= p(1+pq) + q(1-pq) \\
&= p+q.
\end{aligned}$$

(iv) By a similar way of (3), we get  $x(p+q)x = x$ .

Thus,  $(p+q)^\dagger = (1+pq)^\dagger p + q(1-p)$ . □

The next theorem, a main result of this paper, admits proficient skills on  $F$ ,  $G$  and  $H$ , expressing the formulae of the MP-inverse of difference of projectors.

**Theorem 3.9.** *Let  $p - q \in R^\dagger$ . Then*

- (1)  $(1 - pqp)^\dagger = p((p - q)^\dagger)^2 + (1 - p)$ ,
- (2)  $(1 - pq)^\dagger = p((p - q)^\dagger)^2 - pq(p - q)^\dagger + 1 - p$ ,
- (3)  $(p - pqp)^\dagger = p((p - q)^\dagger)^2$ ,
- (4) *If  $p - pq$  is  $*$ -cancellable, then  $(p - pq)^\dagger = (p - q)^\dagger p$ ,*
- (5) *If  $p - pq$  is  $*$ -cancellable, then  $(p - qp)^\dagger = p(p - q)^\dagger$ .*

*Proof.* (1) As  $1 - pqp = p(p - q)^2 + 1 - p$ , then  $(1 - pqp)^\dagger = p((p - q)^\dagger)^2 + 1 - p$  according to Theorem 3.1.

(2) It follows from Theorem 3.3 that  $p - q \in R^\dagger$  implies  $1 - pq \in R^\dagger$ . Let  $x = p((p - q)^\dagger)^2 - pq(p - q)^\dagger + 1 - p$ . We next show that  $x$  is the MP-inverse of  $1 - pq$ .

(i) We have

$$\begin{aligned}
(1 - pq)x &= (1 - pq)[p((p - q)^\dagger)^2 - pq(p - q)^\dagger + 1 - p] \\
&= (p - pqp)((p - q)^\dagger)^2 - (1 - pq)pq(p - q)^\dagger + (1 - pq)(1 - p) \\
&= p(p - q)^2((p - q)^\dagger)^2 - (p - pqp)(p - q)^\dagger(1 - p) + (1 - pq)(1 - p) \\
&= p(p - q)(p - q)^\dagger - p(p - q)^2(p - q)^\dagger(1 - p) + (1 - pq)(1 - p)
\end{aligned}$$

$$\begin{aligned}
&= p(p-q)(p-q)^\dagger - p(p-q)(1-p) + (1-pq)(1-p) \\
&= p(p-q)(p-q)^\dagger + 1-p \\
&= pH + 1-p.
\end{aligned}$$

Hence,  $((1-pq)x)^* = (1-pq)x$  since  $pH = Hp$  and  $H^* = H$ .

(ii) We get  $x(1-pq) = p(p-q)^\dagger p + 1-p$ . Hence,  $(x(1-pq))^* = x(1-pq)$ .

(iii)  $(1-pq)x(1-pq) = 1-pq$ . Indeed,

$$\begin{aligned}
(1-pq)x(1-pq) &= (pH + 1-p)(1-pq) = Hp(1-pq) + (1-p)(1-pq) \\
&= Hp(p-pq) + 1-p = pH(p-pq) + 1-p \\
&= pHp(p-q) + 1-p = pH(p-q) + 1-p \\
&= p(p-q) + 1-p \\
&= 1-pq.
\end{aligned}$$

(iv)  $x(1-pq)x = 1-pq$ . Actually, we can obtain this result by a similar proof of (iii).

(3) Since  $p-pqp = p(p-q)^2 = (p-q)^2p$ , we get  $(p-pqp)^\dagger = p((p-q)^\dagger)^2$  by Lemma 2.2.

(4) Keeping in mind that  $a^\dagger = a^*(aa^*)^\dagger = (a^*a)^\dagger a^*$ , we have  $(p-pq)^\dagger = (p-qp)p((p-q)^\dagger)^2 = (p-q)((p-q)^\dagger)^2p = (p-q)^\dagger p$ .

(5) Note that  $a$  is  $*$ -cancellable if and only if  $a^*$  is  $*$ -cancellable. It follows from  $(a^*)^\dagger = (a^\dagger)^*$  that  $(p-qp)^\dagger = p(p-q)^\dagger$ .  $\square$

**Corollary 3.10.** *Let  $p-pq$  be  $*$ -cancellable and let  $1-pq \in R^\dagger$ . Then  $p-q \in R^\dagger$  and*

$$(p-q)^\dagger = (1-pq)^\dagger(p-pq) + (p+q-pq)^\dagger(pq-q).$$

*Proof.* From Theorem 3.3, we have  $p-q \in R^\dagger \Leftrightarrow 1-pq \in R^\dagger$ .

By Theorem 3.9 (2), we have  $(p+q-pq)^\dagger = (1-p)((p-q)^\dagger)^2 + (1-p)(1-q)(p-q)^\dagger + p$ . It is straight to check that  $(1-pq)^\dagger(p-pq) + (p+q-pq)^\dagger(pq-q)$  satisfies four equations of Penrose.  $\square$

The following result is motivated by [2], therein, Deng considered the Drazin inverses of difference of idempotent bounded operators on Hilbert spaces.

**Theorem 3.11.** *Let  $pq - qp$  be  $*$ -cancellable. Then*

- (1)  $(p - q)^\dagger = p - q$  if and only if  $pq = qp$ ,
- (2) If 6 is invertible in  $R$ , then  $(p + q)^\dagger = p + q$  if and only if  $pq = 0$ .

*Proof.* (1) If  $pq = qp$ , it is straightforward to check  $(p - q)^\dagger = p - q$ .

Conversely,  $(p - q)^\dagger = p - q$  implies  $(p - q)^3 = p - q$ , we get  $pqp = qpq$  and hence  $(pq - qp)^*(pq - qp) = 0$ . It follows that  $pq = qp$  since  $pq - qp$  is  $*$ -cancellable.

(2) Suppose  $pq = 0$ . Then  $p^*q = pq^* = 0$  since  $p, q$  are projectors. Then  $(p + q)^\dagger = p + q$  by Lemma 2.3.

Conversely,  $(p + q)^\dagger = p + q$  concludes  $(p + q)^3 = p + q$ . By direct calculations, it follows that  $2pq + 2qp + pqp + qpq = 0$ . (3.1)

Multiplying the equality (3.1) by  $p$  on the left yields  $2pq + 3pqp + pqpq = 0$ . (3.2)

Multiplying the equality (3.1) by  $q$  on the right gives  $2pq + 3qpq + pqpq = 0$ . (3.3)

Combining the equalities (3.2) and (3.3), it follows that  $pqp = qpq$  since 3 is invertible. As  $pq - qp$  is  $*$ -cancellable, then  $pqp = qpq$  implies  $pq = qp$ . Hence, equality (3.1) can be reduced to  $6pq = 0$ .

Thus,  $pq = 0$ . □

**Theorem 3.12.** *Let  $1 - p - q \in R^\dagger$ . Then*

- (1)  $pqp \in R^\dagger$  and  $(pqp)^\dagger = p((1 - p - q)^\dagger)^2 = ((1 - p - q)^\dagger)^2 p$ ,
- (2) If  $pq$  is  $*$ -cancellable, then  $pq \in R^\dagger$  and  $(pq)^\dagger = qp((1 - p - q)^\dagger)^2$ .

*Proof.* (1) Since  $(1 - p - q)^* = 1 - p - q$ , we have  $((1 - p - q)^2)^\dagger = ((1 - p - q)^\dagger)^2$  by Lemma 2.2. As  $pqp = p(1 - p - q)^2 = (1 - p - q)^2 p$ , then  $pqp \in R^\dagger$  from Lemma 2.2 and hence  $(pqp)^\dagger = p((1 - p - q)^\dagger)^2 = ((1 - p - q)^\dagger)^2 p$ .

(2) Note that  $1 - p - q \in R^\dagger$  implies  $pqp \in R^\dagger$ . As  $pqp = pq(pq)^*$  and  $pq$  is  $*$ -cancellable, then  $pq \in R^\dagger$  by [8, Theorem 5.4]. The formula  $a^\dagger = a^*(aa^*)^\dagger$  guarantees that  $(pq)^\dagger = qp((1 - p - q)^\dagger)^2$ . □

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